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Fractional Electromagnetic Equations Using Fractional Forms

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Abstract The generalized physics laws involving fractional derivatives give new models and conceptions that can be used in complex systems having memory effects. Using the fractional differential forms, the classical electromagnetic equations involving the fractional derivatives have been worked out. The fractional conservation law for the electric charge and the wave equations were derived by using this method. In addition, the fractional vector and scalar potentials and the fractional Poynting theorem have been derived.

Keywords Fractional differential forms · Fractional Caputo derivatives · Fractional Maxwell's equations · Fractional Poynting theorem · Fractional vector potential

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1 Introduction

Fractional derivatives and integrals are old topics. Although they have long mathematical history, they have just found application in science [15, 17, 18, 25, 26, 28, 29, 35, 39, 40]. But in a fairly short period of time the list of such applications becomes so many. For example, they includes branches of mechanics and physics as chaotic dynamics, quantum mechanics, plasma physics, anomalous diffusion, etc. [15, 35, 39, 40]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivative in comparison with classical integer order. Using models involving local scaling properties the fractional derivatives were renormalized to construct local fractional differential operators [19, 20]. In mechanics Riewe has shown that Lagrangian involving fractional time derivatives leads to equation of motion with non conservative classical derivatives such as friction [33, 34]. During the last decades several researchers have explored in this area and the gave new insight into this problem [1–8, 14, 16, 21–24, 27, 30–32, 36, 37, 41]. Differential forms have many application in physics [13]. Fractional generalization of differential forms has been presented in [9, 10]. Furthermore, a fractional generalization of differential forms has been suggested in [11, 12]. The application of fractional differential forms to dynamical systems are considered in [37].

The aim of this study is to extend the fractional forms to the electromagnetism. We mention that some ways were suggested by authors to fractionalize the electromagnetic equations [38].

This paper is organized as follows: Sect. 2 gives a brief definition of the Caputo fractional derivative together with the definition of the classical and the fractional forms and their exterior derivatives. Section 3 contains the electromagnetic equations in the language of calculus of classical forms. It is shown that the electromagnetic equations arrived as the result of applying the exterior derivative on determined forms. The Sect. 4 contains the generalization of the electromagnetic equations versus the classical forms obtained in the last section to fractional one. In Sect. 5 we develop the fractional classical conservative law for electric charge. Section 6 deals with fractional Poynting's theorem. In Sect. 7 we defined the fractional vector and the scalar potentials together with their equations. The fractional wave equations were derived in Sect. 8. Finally, Sect. 9 is devoted to our conclusions.

2 Fractional Differential Forms

The calculus of classical differential forms is a powerful tool in applied mathematics. There are so many books that give a clear introduction to this field [13]. In calculus when a new function appears in the scene, it is natural to ask what its derivative is. Similarly with form, it is reasonable to ask what its exterior derivative is. For example a 1-form, integer order one, can be shown as follows

$$\omega = \sum_{i=1}^{n} a_i dx_i. \tag{1}$$

The classical exterior derivative is defined as

$$d = dx_i \frac{\partial}{\partial x_i}. \tag{2}$$

In [11, 12] the author generalizes the definition of integer order vector spaces form to fractional order one, and denotes it by $F(\nu, m, n)$. In this notation ν is the order of differential form, m the number of coordinate differential appearing in the basis elements, n the number of coordinates. For instance (1) is an element of $F(1, 1, n)$.

The definition of fractional exterior derivative: If the partial derivative in the definition of the classical exterior derivative, is replaced by the fractional order, we lead to definition of fractional exterior derivative,

$$d^\nu = \sum_{i=1}^{n} dx_i^\nu \ {}_0^C D_{x_i}^\nu, \tag{3}$$

where ${}_0^C D_x^\nu$ is left Caputo fractional derivative that is defined as,

$${}_0^C D_x^\nu f = \frac{1}{\Gamma(m-\nu)} \int_0^x (x-y)^{m-\nu-1} \frac{\partial^m f}{\partial y^m} dy, \tag{4}$$

and $\nu > 0$ and m is the first whole number greater than or equal to ν.
Let $\sigma \in F(\nu, 1, n)$,

$$\sigma = \sum_{i=1}^{n} \sigma_i dx_i^\nu, \tag{5}$$

and consider its fractional exterior derivative,

$$d^\nu \sigma = \sum_{i=1}^{n} d^\nu (\sigma_i dx_i^\nu). \tag{6}$$

Using the product rule of exterior fractional derivative we have

$$d^\nu \sigma = \sum_{i=1}^{n} \sum_{j=1}^{n} {}_0^C D_{xj}^\nu \sigma_i dx_j^\nu \wedge dx_i^\nu. \tag{7}$$

If $d^\nu \sigma = 0$ we have

$${}_0^C D_{xj}^\nu \sigma_i - {}_0^C D_{xi}^\nu \sigma_j = 0. \tag{8}$$

Note that in the following equations we will omit the $\wedge$ sign between the differential forms. Also we will show ${}_0^C D_{xj}^\nu$ simply D_{xj}^ν. The fractional gradient, divergence, and curl are defined as follows, respectively:

$$\mathtt{grad}^\nu f = \mathbf{x} \ {}_0^C D_x^\nu f + \mathbf{y} \ {}_0^C D_y^\nu f + \mathbf{z} \ {}_0^C D_z^\nu f, \tag{9}$$

$$\mathtt{div}^\nu \mathbf{F} = {}_0^C D_x^\nu F_x + {}_0^C D_y^\nu F_y + {}_0^C D_z^\nu F_z, \tag{10}$$

and

$$\mathtt{curl}^\nu \mathbf{F} = \mathbf{x}({}_0^C D_y^\nu F_z - {}_0^C D_z^\nu F_y) + \mathbf{y}({}_0^C D_z^\nu F_x - {}_0^C D_x^\nu F_z) + \mathbf{z}({}_0^C D_x^\nu F_y - {}_0^C D_y^\nu F_x), \tag{11}$$

where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are unit vectors.

3 Maxwell Equations of Integer Order Form

In classical electromagnetic field theory one deals with the following quantities:

$\mathbf{E}$ = electric field,	$\mathbf{H}$ = magnetic field,
$\mathbf{B}$ = magnetic induction,	$\mathbf{J}$ = electric current density,
$\mathbf{D}$ = dialectic displacement,	ρ = charge density.

All of them are functions of space variables x_1, x_2, x_3 and the time t. The basic Maxwell equations in ordinary vector language are Faraday's law induction

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t}, \tag{12}$$

Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{B}}{\partial t}, \tag{13}$$

continuity equation

$$\nabla \cdot \mathbf{D} = \rho, \tag{14}$$

non-existence of monopole magnetic

$$\nabla \cdot \mathbf{B} = 0. \tag{15}$$

Also classical wave equations

$$v^2 \nabla^2 \mathbf{B} - \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0, \tag{16}$$

where v is the velocity of wave, and

$$v^2 \nabla^2 \mathbf{E} - \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{17}$$

We shall put these equation into language of exterior forms [13]. To this end we set

$$\alpha = (E_1 dx_1 + E_2 dx_2 + E_3 dx_3)dt + B_1 dx_2 dx_3 + B_2 dx_3 dx_1 + B_3 dx_1 dx_2 \tag{18}$$

and

$$\beta = -(H_1 dx_1 dt + H_2 dx_2 dt + H_3 dx_3 dt) + D_1 dx_2 dx_3 + D_2 dx_3 dx_2 + D_3 dx_1 dx_2 \tag{19}$$

and

$$\gamma = (J_1 dx_2 dx_3 + J_2 dx_3 dx_1 + J_3 dx_1 dx_3)dt - \rho dx_1 dx_2 dx_3. \tag{20}$$

In is important to say that these quantities are two-form. Equations (12) and (15) are obtained by applying the classical exterior derivative on α,

$$d\alpha = 0, \tag{21}$$

where d is classical exterior derivative and defined as

$$d = \frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \frac{\partial}{\partial x_3} dx_3 + \frac{\partial}{\partial t} dt. \tag{22}$$

Also (13) and (14) become

$$d\beta + c\gamma = 0. \tag{23}$$

Applying d to this last equation yields

$$d\gamma = 0. \tag{24}$$

In vector notation this equation is

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{25}$$

In this section the Maxwell equations are written in language of differential form and exterior derivative. In the next section we will generalize these equations.

4 Fractional Maxwell Equation

In order to fractionalize the electromagnetic equations, let us consider the Maxwell's equations in the previous section. Then we generalize the determined forms to fractional forms and apply the fractional exterior derivative we have obtain the fractional Maxwell's equation as follows [36]

$$\alpha^\nu = (E_1 dx_1^\nu + E_2 dx_2^\nu + E_3 dx_3^\nu)dt^\nu + B_1 dx_2^\nu dx_3^\nu + B_2 dx_3^\nu dx_1^\nu + B_3 dx_1^\nu dx_2^\nu, \tag{26}$$

$\alpha^\nu \in F(\nu, 2, 4)$. The fractional exterior derivative for this form is defined as

$$d^\nu = D_{x_1}^\nu dx_1^\nu + D_{x_2}^\nu dx_2^\nu + D_{x_3}^\nu dx_3^\nu + D_t^\nu dt^\nu. \tag{27}$$

Applying it on α^ν we obtain an element of $F(2\nu, 3, 4)$. Like the classical case if we take the result zero, we lead to the fractional electromagnetic equations

$$d^\nu \alpha^\nu = 0. \tag{28}$$

In component notation,

$$\{D_{x_1}^\nu E_2 - D_{x_2}^\nu E_1\} dx_1^\nu dx_2^\nu dt^\nu + \{D_{x_1}^\nu E_3 - D_{x_3}^\nu E_1\} dx_1^\nu dx_3^\nu dt^\nu \tag{29}$$

$$+ \{D_{x_3}^\nu E_2 - D_{x_2}^\nu E_3\} dx_3^\nu dx_2^\nu dt^\nu = -D_t^\nu B_3 dx_1^\nu dx_2^\nu dt^\nu \tag{30}$$

$$- D_t^\nu B_2 dx_1^\nu dx_3^\nu dt^\nu - D_t^\nu B_1 dx_3^\nu dx_2^\nu dt^\nu, \tag{31}$$

and,

$$\{D_{x_1}^\nu B_1 + D_{x_2}^\nu B_2 + D_{x_2}^\nu B_3\} dx_1^\nu dx_3^\nu dx_3^\nu = 0. \tag{32}$$

In vector notation we obtain

$$\mathtt{curl}^\nu \mathbf{E} = -D_t^\nu \mathbf{B}, \tag{33}$$

and

$$\mathtt{div}^\nu \mathbf{B} = 0. \tag{34}$$

We generalize β as follows, namely

$$\beta = -(H_1 dx_1^\nu dt^\nu + H_2 dx_2^\nu dt^\nu + H_3 dx_3^\nu dt^\nu) + D_1 dx_2^\nu dx_3^\nu + D_2 dx_3^\nu dx_2^\nu + D_3 dx_1^\nu dx_2^\nu, \quad (35)$$

in similar to classical case,

$$d^\nu \beta^\nu + \gamma^\nu = 0, \quad (36)$$

thus, the fractional equations are given

$$\mathtt{curl}^\nu \mathbf{H} = J + D_t^\nu \mathbf{D}, \quad (37)$$

$$\mathtt{div}^\nu \mathbf{D} = \rho. \quad (38)$$

5 Fractional Conservation Law for Electric Charge

In classical electromagnetic if we apply the exterior derivative, d, on γ, we have

$$d\gamma = 0, \quad (39)$$

that after some manipulation we lead to equation of classical conservative law of electric charge,

$$\mathtt{div}\mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad (40)$$

Now we generalize the equation of classical conservative law of electric charge, (39) to fractional one as given below,

$$d^\nu \gamma^\nu = 0. \quad (41)$$

After some simple algebraic calculation we obtain the following

$$\{D_{x_1}^\nu J_1 + D_{x_2}^\nu J_2 + D_{x_3}^\nu J_3 + D_t^\nu \rho\} dx_1^\nu dx_2^\nu dx_3^\nu dt^\nu = 0. \quad (42)$$

If we suppose

$$\mathtt{div}^\nu = (D_{x_1}^\nu, D_{x_2}^\nu, D_{x_3}^\nu), \quad (43)$$

we obtain

$$\mathtt{div}^\nu \mathbf{J} + D_t^\nu \rho = 0. \quad (44)$$

This equation is called fractional conservation law for electric charge.

6 Fractional Poynting Energy Flux S

We can reformulate fractional Maxwell equations in new fractional 1-forms and 2-forms as

$$\omega_1^\nu = E_1 dx_1^\nu + E_2 dx_2^\nu + E_3 dx_3^\nu, \quad (45)$$

$$\omega_2^\nu = B_1 dx_2^\nu dx_3^\nu + B_2 dx_3^\nu dx_1^\nu + B_3 dx_1^\nu dx_2^\nu, \quad (46)$$

$$\omega_3^\nu = H_1 dx_1^\nu + H_2 dx_2^\nu + H_3 dx_3^\nu, \tag{47}$$

$$\omega_4^\nu = D_1 dx_2^\nu dx_3^\nu + D_2 dx_3^\nu dx_1^\nu + D_3 dx_1^\nu dx_2^\nu, \tag{48}$$

$$\omega_5^\nu = J_1 dx_2^\nu dx_3^\nu + J_2 dx_3^\nu dx_1^\nu + J_3 dx_1^\nu dx_2^\nu, \tag{49}$$

and if we decompose the total exterior derivative to two part, space part, d_s^ν, and time part, d_t^ν, the fractional Maxwell equations can express as follows, Faraday's law induction,

$$d_s^\nu \omega_1 = -d_t^\nu \omega_2, \tag{50}$$

Ampere's law

$$d_s^\nu \omega_3 = \omega_5 + d_t^\nu \omega_4, \tag{51}$$

nonexistence of monopole

$$d_s^\nu \omega_2 = 0, \tag{52}$$

continuity of charge

$$d_s^\nu \omega_4 = \rho dx_1^\nu d_2^\nu dx_3^\nu. \tag{53}$$

This approach helps us to define fractional Poynting theorem as follows:
Let us define S^ν as

$$S^\nu = \omega_1^\nu \wedge \omega_2^\nu = S_1 dx_2^\nu dx_3^\nu + S_2 dx_3^\nu dx_1^\nu + S_3 dx_1^\nu dx_2^\nu. \tag{54}$$

Let us to consider ω_1 and ω_3 as

$$\omega_1^\nu = \sum_{i=1}^{3} E_i dx_i^\nu, \qquad \omega_3^\nu = \sum_{j=1}^{3} H_j dx_j^\nu, \tag{55}$$

then we have

$$\omega_1^\nu \wedge \omega_3^\nu = \sum_{i=1}^{3}\sum_{j=1}^{3} E_i H_j dx_i^\nu dx_j^\nu. \tag{56}$$

Applying d^ν

$$d^\nu\{\omega_1^\nu \wedge \omega_3^\nu\} = \sum_{k=1}^{3} D_{x_k}^\nu (E_i H_j) dx_k^\nu dx_i^\nu dx_j^\nu, \tag{57}$$

by using fractional rule of product we lead

$$D_{x_k}^\nu (E_i H_j) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=0}^{\infty} \binom{\nu}{m} D_{x_k}^{\nu-m} E_i \frac{\partial^m H_j}{\partial x_{k^m}} dx_k^\nu dx_i^\nu dx_j^\nu. \tag{58}$$

By using the definition of fractional divergence, $\mathtt{div}^\nu$, we have

$$\mathtt{div}^\nu S^\nu = -\omega_2^\nu \wedge d_t^\nu \omega_2 + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{\infty} \binom{\nu}{m} D_{x_k}^{\nu-m} E_i \frac{\partial^m H_j}{\partial x_k^m} dx_k^\nu dx_i^\nu dx_j^\nu. \tag{59}$$

This is called the theorem of fractional Poynting.

7 Fractional Vector Potential A

$\alpha \in F(2\nu, 2, 4)$ and $d^\nu \alpha = 0$. If α be exact form then there exist $\lambda \in F(\nu, 1, 4)$ such that $d^\nu \lambda = \alpha$ then if we denote A as

$$\lambda = A_1 dx_1^\nu + A_2 dx_2^\nu + A_3 dx_3^\nu + \varphi d^\nu t, \tag{60}$$

and applying d^ν on λ we lead

$$\mathtt{curl}^\nu \mathbf{A} = \mathbf{B}, \tag{61}$$

and

$$\mathtt{grad}^\nu \varphi - D_t^\nu \mathbf{A} = \mathbf{E}. \tag{62}$$

From (37), and assumption of free space, $\mathbf{B} = \mu \mathbf{H}$, $\mathbf{D} = \epsilon \mathbf{E}$, $v^2 = \frac{1}{\epsilon\mu}$ we have

$$v^2 \mathtt{curl}^\nu \mathbf{B} = D_t^\nu \mathbf{E}. \tag{63}$$

If we replace from (61) and (62) we have

$$\mathtt{curl}^\nu \mathtt{curl}^\nu A = D_t^\nu \{\mathtt{grad}^\nu \varphi - D_t^\nu \mathbf{A}\}. \tag{64}$$

Using the following relation

$$\mathtt{curl}^\nu \mathtt{curl}^\nu \mathbf{F} = \mathtt{grad}^\nu \mathtt{div}^\nu \mathbf{F} - \mathtt{div}^\nu \mathtt{grad}^\nu \mathbf{F}, \tag{65}$$

and by using the notation $\mathtt{div}^\nu \mathtt{grad}^\nu =< D^\nu, D^\nu >= (D^\nu)^2$, s the spacial part and t is temporal part of D^ν, and $D_t^\nu D_t^\nu = D_t^{2\nu}$ means that the D_t^ν 2-times operates we have

$$\mathtt{grad}^\nu \mathtt{div}^\nu \mathbf{A} - (D_s^\nu)^2 \mathbf{A} = \mathtt{grad}^\nu D_t^\nu \varphi - D_t^{2\nu} \mathbf{A}. \tag{66}$$

The fractional Lorentz condition becomes

$$\mathtt{div}^\nu \mathbf{A} - D_t^\nu \varphi = 0. \tag{67}$$

If the Lorentz condition is be satisfied, then

$$D_t^{2\nu} \mathbf{A} - (D_s^\nu)^2 \mathbf{A} = 0. \tag{68}$$

8 Fractional Wave Equations

By using the time fractional derivation formula (37), the fractional Ampere's law, and by substitution (33), using $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \epsilon \mathbf{E}$, $v^2 = \frac{1}{\epsilon\mu}$, we finally obtain

$$v^2 \mathtt{curl}^\nu \mathtt{curl}^\nu \mathbf{E} = D_t^{2\nu} \mathbf{E}. \tag{69}$$

By using (65) we get

$$\mathtt{curl}^\nu \mathtt{curl}^\nu \mathbf{E} = \mathtt{grad}^\nu \mathtt{div}^\nu \mathbf{E} - (D_s^\nu)^2 \mathbf{E}. \tag{70}$$

For free space $\rho = 0$, $\mathbf{J} = 0$, then we have

$$\mathtt{div}^{\nu}\mathbf{E} = 0, \tag{71}$$

thus, the wave equation,

$$v^2(D_s^{\nu})^2\mathbf{E} - D_t^{2\nu}\mathbf{E} = 0. \tag{72}$$

By time differentiation formula (33), and substitution (37) we have

$$v^2\mathtt{crul}^{\nu}\mathtt{crul}^{\nu}\mathbf{B} = D_t^{2\nu}\mathbf{B}, \tag{73}$$

therefore

$$\mathtt{curl}^{\nu}\mathtt{curl}^{\nu}\mathbf{B} = \mathtt{grad}^{\nu}\mathtt{div}^{\nu}\mathbf{B} - (D_s^{\nu})^2\mathbf{B}. \tag{74}$$

From fractional Maxwell equation we have $\mathtt{div}^{\nu}\mathbf{B} = 0$, thus we conclude that

$$v^2(D_s^{\nu})^2\mathbf{B} - D_t^{2\nu}\mathbf{B} = 0, \tag{75}$$

is fractional wave equation of $\mathbf{B}$.

9 Conclusion

In recent years the application of fractional derivative and integral started to be used in various fields. For example in mechanics, the fractional mechanics describes both conservative and non-conservative systems. On the other hand the fractional mechanics is a non-local theory. Fractional derivatives are also applicable on materials that have electromagnetic memory properties. We have used fractional time derivative as well as fractional space derivative in this work. In this work at first we quote the Maxwell equations in the language of differential forms, and after that,by using the fractional differential forms, the fractional Maxwell equations were obtained. Further, we have explained fractional wave equations, fractional vector and scalar potentials and fractional Poynting theorem.

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